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ON WAVE PROPAGATION IN INEXTENSIBLE ELASTIC BODIES

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Abstract—In this paper we study the propagation of acceleration waves in inextensible elastic bodies.* While the computations are but an exercise, the results are interesting and quite unlike the corresponding results for unconstrained bodies. Indeed, a wave travelling in the direction of inextensibility must necessarily be transverse, and, when the reaction stress is compressive and sufficiently large, the corresponding speed of propagation becomes non-real, so that even transverse waves fail to exist.

We also study (infinitesimal) progressive waves, and we find the corresponding propagation condition to be the same as that for acceleration waves. Here, however, non-real speeds of propagation have a definite physical meaning: they imply exponential growth of the wave. Thus, in particular, when the reaction stress is compressive and sufficiently large, a transverse progressive wave travelling in the direction of inextensibility grows without bound. We conjecture that this indicates the presence of local buckling.†

1. INEXTENSIBLE ELASTIC BODIES

For convenience, we identify the body $\mathscr{B}$ with the region of space it occupies in a fixed reference configuration with density $\rho(\mathbf{X})$. A motion of $\mathscr{B}$ is then a mapping $\mathbf{x}$; its value $\mathbf{x}(\mathbf{X}, t)$ is the position of the material point $\mathbf{X}$ at time t.

We assume that the body is inextensible in the direction $\mathbf{e}$ (in the reference configuration), where $\mathbf{e}$ is a unit vector:

$$|\mathbf{e}| = 1. \tag{1.1}$$

Then every motion of $\mathscr{B}$ obeys the constraint

$$|\mathbf{F}\mathbf{e}| = 1, \tag{1.2}$$

where

$$\mathbf{F} = \nabla\mathbf{x} \tag{1.3}$$

* The constraint of inextensibility was apparently first considered by Adkins and Rivlin [1].

† Truesdell and Noll [2], p. 275, in a discussion of non-real speeds of propagation for infinitesimal progressive waves, remark that: "Perhaps in this way physical instabilities such as buckling ultimately may come to be explained. It may be the occurrence of certain waves, presumably transverse, that disturbs a sufficiently severe homogeneous compression of a bar so much as to carry the body over into a state of small oscillation about a different configuration subject to the same resultant terminal loads." See also Truesdell [3], §4.

is the deformation gradient. Further, for an elastic body the Piola–Kirchhoff stress $\mathbf{S}$ is determined by $\mathbf{F}$ only to within an arbitrary pure tension* σ

$$\mathbf{S} = \sigma(\mathbf{Fe} \otimes \mathbf{e}) + \hat{\mathbf{S}}(\mathbf{F}) \tag{1.4}$$

with $\hat{\mathbf{S}}$ a smooth function. Of course, $\hat{\mathbf{S}}$ also depends explicitly on the material point $\mathbf{X}$, but, for convenience, we suppress this dependence.

2. ACCELERATION WAVES

Assume now that an acceleration wave Σ exists in the body. Across Σ the motion $\mathbf{x}$, the velocity $\dot{\mathbf{x}}$, the deformation gradient $\mathbf{F}$, and the reaction stress σ are continuous, but $\ddot{\mathbf{x}}$, $\dot{\mathbf{F}}$, $\nabla\mathbf{F}$, $\dot{\sigma}$, and $\nabla\sigma$ suffer jump discontinuities. We call

$$\mathbf{a} = [\ddot{\mathbf{x}}] \tag{2.1}$$

the amplitude; here we have used the standard notation $[f]$ for the jump in a function f across Σ.† Then, as is well known, we have the kinematical condition of compatibility‡

$$[\dot{\mathbf{F}}] = -\frac{1}{U}\mathbf{a} \otimes \mathbf{n}, \tag{2.2}$$

as well as the following expression for balance of momentum across the wave§

$$[\dot{\mathbf{S}}]\mathbf{n} = -\rho U \mathbf{a}. \tag{2.3}$$

Here U is the speed of propagation and $\mathbf{n}$ the direction of propagation, both relative to the reference configuration.

If we differentiate (1.2) (in the form $\mathbf{Fe} \cdot \mathbf{Fe} = 1$) with respect to time and evaluate the resulting relation across Σ, we arrive at the result

$$(\mathbf{a} \cdot \mathbf{f})(\mathbf{n} \cdot \mathbf{e}) = 0, \tag{2.4}$$

where

$$\mathbf{f} = \mathbf{Fe} \tag{2.5}$$

is the direction of inextensibility relative to the current configuration. By (2.4) and (2.5),

$$\mathbf{n} = \mathbf{e} \Rightarrow \mathbf{Fn} \cdot \mathbf{a} = 0. \tag{2.6}$$

Since $\mathbf{Fn}$ is the direction of propagation of the wave relative to the current configuration, (2.6) asserts that a wave travelling in the direction of inextensibility must necessarily be transverse.

Next, (2.2) and (2.5) imply that

$$[\dot{\mathbf{f}}] = -\frac{1}{U}(\mathbf{n} \cdot \mathbf{e})\mathbf{a}, \tag{2.7}$$

* See, e.g. Truesdell and Noll [2], p. 72; Gurtin and Podio Guidugli [4]. Note that the Piola–Kirchhoff reaction stress $\sigma(\mathbf{Fe} \otimes \mathbf{e})$ is equivalent to the Cauchy reaction stress $\sigma J^{-1}(\mathbf{Fe} \otimes \mathbf{Fe})$, where $J = \det \mathbf{F}$. Thus, more precisely, σJ^{-1} rather than σ is the pure tension due to the reaction.

† See, e.g. Truesdell and Toupin [5], §173.

‡ See, e.g. [5], Eq. (190.2); our U and $\mathbf{n}$ are designated by U_N and $\mathbf{N}$, respectively, in [5].

§ See, e.g. Chen [6], p. 314; Gurtin [7], p. 254.

so that the jump in $\dot{\mathbf{f}}$ is parallel to the amplitude. If we take the time-derivative of (1.4), we conclude, with the aid of (2.2), (2.3), (2.5), and (2.7) that

$$\sigma(\mathbf{n}\cdot\mathbf{e})^2\mathbf{a} - U[\dot{\sigma}](\mathbf{n}\cdot\mathbf{e})\mathbf{f} + \mathbf{Q}(\mathbf{n})\mathbf{a} = \rho U^2\mathbf{a}. \tag{2.8}$$

Here $\mathbf{Q}(\mathbf{n})$ is the *acoustic tensor*; that is, $\mathbf{Q}(\mathbf{n})$ is the unique tensor with the following property:

$$\mathbf{Q}(\mathbf{n})\mathbf{v} = \hat{\mathbf{S}}_{\mathbf{F}}(\mathbf{F})[\mathbf{v}\otimes\mathbf{n}]\mathbf{n} \tag{2.9}$$

for every vector $\mathbf{v}$, where $\hat{\mathbf{S}}_{\mathbf{F}}$ is the derivative of $\hat{\mathbf{S}}$ with respect to $\mathbf{F}$.

In view of (2.4), either $\mathbf{n}$ is perpendicular to $\mathbf{e}$, or $\mathbf{a}$ is perpendicular to $\mathbf{f}$. We now consider these two cases separately.

Case 1

$$\mathbf{n}\cdot\mathbf{e} = 0. \tag{2.10}$$

Here, trivially, (2.8) implies that

$$\mathbf{Q}(\mathbf{n})\mathbf{a} = \rho U^2\mathbf{a}. \tag{2.11}$$

Thus the classical Fresnel–Hadamard propagation condition holds when the wave is propagating in a direction perpendicular to the direction of inextensibility.

Case 2

$$\mathbf{f}\cdot\mathbf{a} = 0. \tag{2.12}$$

Since

$$|\mathbf{f}| = 1, \tag{2.13}$$

as is clear from (1.2) and (2.5), if we take the inner product of (2.8) with $\mathbf{f}$ and use (2.12), we arrive at the result

$$U[\dot{\sigma}](\mathbf{n}\cdot\mathbf{e}) = \mathbf{f}\cdot\mathbf{Q}(\mathbf{n})\mathbf{a}. \tag{2.14}$$

This result, when combined with (2.8), leads to the propagation condition

$$\mathbf{P}\mathbf{Q}(\mathbf{n})\mathbf{a} = [\rho U^2 - \sigma(\mathbf{n}\cdot\mathbf{e})^2]\mathbf{a}. \tag{2.15}$$

Here

$$\mathbf{P} = \mathbf{1} - \mathbf{f}\otimes\mathbf{f} \tag{2.16}$$

is the projection onto the plane perpendicular to $\mathbf{f}$; its presence in (2.15) insures that solutions $\mathbf{a}$ of (2.15) are consistent with (2.12).

When the reaction stress σ is zero, (2.15) reduces to

$$\mathbf{P}\mathbf{Q}(\mathbf{n})\mathbf{a}_0 = \rho U_0{}^2\mathbf{a}_0, \tag{2.17}$$

and it is clear that there exists a one-to-one correspondence between solutions $(U_0, \mathbf{a}_0)$ of (2.17) and solutions $(U, \mathbf{a})$ of (2.15); in fact, this correspondence is given by

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_0, \\ \rho U^2 &= \rho U_0{}^2 + \sigma(\mathbf{n}\cdot\mathbf{e})^2. \end{aligned} \tag{2.18}$$

If we assume that $\mathbf{n}\cdot\mathbf{e} \neq 0$, then a tensile reaction stress raises the speed of propagation, a compressive reaction stress lowers the speed of propagation. Further, given any $\mathbf{n} \neq \mathbf{e}$, when the reaction stress σ is compressive and sufficiently large, it is impossible to propagate a wave in the direction of $\mathbf{n}$, for in this instance the corresponding speeds U will all be imaginary.

The existence of imaginary speeds leads one to conjecture the presence of a local instability; the fact that this phenomenon occurs at high compressive stresses makes this conjecture intuitively plausible. With this in mind, we study, in the next section, infinitesimal progressive waves for which imaginary propagation speeds actually signify an instability.

3. INFINITESIMAL PROGRESSIVE WAVES

In order to investigate the stability of the material when the compressive stress is compressive and large, we now turn to a study of infinitesimal progressive waves. Such waves are solutions of the linearized field equations obtained under the approximative assumption that

$$\varepsilon = |\nabla \mathbf{u}| + |\sigma - \sigma_0| \tag{3.1}$$

is small. Here

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \tag{3.2}$$

is the displacement, while $\sigma_0 \neq 0$ is a prescribed constant reaction stress. If in (1.2) and (1.4) terms of order ε^2 are neglected, then these equations reduce to*

$$\begin{gathered} \mathbf{e} \cdot \mathbf{H}\mathbf{e} = 0, \\ \mathbf{S} = (\sigma - \sigma_0)\mathbf{e} \otimes \mathbf{e} + \sigma_0 \mathbf{H}\mathbf{e} \otimes \mathbf{e} + \mathsf{C}[\mathbf{H}], \end{gathered} \tag{3.3}$$

where

$$\mathbf{H} = \nabla \mathbf{u} \tag{3.4}$$

is the displacement gradient, and

$$\mathsf{C} = \hat{\mathbf{S}}_{\mathbf{F}}(\mathbf{1}) \tag{3.5}$$

is the elasticity tensor, and where now $\mathbf{S}$ denotes the actual stress minus the constant residual stress $\sigma_0 \mathbf{e} \otimes \mathbf{e}$. To these equations we adjoin the equation of motion

$$\operatorname{Div} \mathbf{S} = \rho \ddot{\mathbf{u}}. \tag{3.6}$$

We now assume that both the body and the reference configuration are homogeneous, so that ρ and C are independent of $\mathbf{X}$. An infinitesimal progressive wave is a solution of (3.3) and (3.6) of the form†

$$\begin{gathered} \mathbf{u}(\mathbf{X}, t) = \mathbf{a} \exp\left\{\frac{i\omega}{U}(\mathbf{p} \cdot \mathbf{n} - Ut)\right\}, \\ \sigma(\mathbf{X}, t) - \sigma_0 = \alpha \exp\left\{\frac{i\omega}{U}(\mathbf{p} \cdot \mathbf{n} - Ut)\right\}, \\ \mathbf{p} = \mathbf{X} - \mathbf{X}_0; \end{gathered} \tag{3.7}$$

U is the speed, $\mathbf{n}$ is the direction, ω is the frequency, and $\mathbf{a}$ is the amplitude. Then, letting

$$\beta = \exp\left\{\frac{i\omega}{U}(\mathbf{p} \cdot \mathbf{n} - Ut)\right\}, \tag{3.8}$$

* These equations are derived by Gurtin and Podio Guidugli [8]. Material frame-indifference implies that $\mathsf{C}[\nabla \mathbf{u}] = \mathsf{C}[\frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)]$, but this fact is not needed in what follows.

† Here we allow complex solutions; these are to be interpreted in the usual manner.

we see that

$$\mathbf{H} = \frac{i\omega}{U}\mathbf{a}\otimes\mathbf{n}\beta, \qquad \dot{\mathbf{H}} = \frac{\omega^2}{U}\mathbf{a}\otimes\mathbf{n}\beta,$$
$$\ddot{\mathbf{u}} = -\omega^2\mathbf{a}\beta, \qquad \dot{\sigma} = -i\omega\alpha\beta, \tag{3.9}$$

and $(3.3)_1$, $(3.9)_1$ imply that

$$(\mathbf{a}\cdot\mathbf{e})(\mathbf{n}\cdot\mathbf{e}) = 0. \tag{3.10}$$

Thus, as before, a wave traveling in the direction of inextensibility must necessarily be transverse.

Next, we conclude from $(3.3)_2$, $(3.7)_2$, (3.8), and $(3.9)_1$ that **S** has the form

$$S = \mathbf{A}\beta \tag{3.11}$$

with **A** a constant tensor (which in general will be complex). A simple calculation, based on (3.11), tells us that $\dot{\mathbf{S}}\mathbf{n} = -U \operatorname{Div}\mathbf{S}$, and therefore $(3.3)_2$, (3.6), and (3.9) imply that

$$-\frac{iU(\mathbf{n}\cdot\mathbf{e})\alpha}{\omega}\mathbf{e} + \sigma_0(\mathbf{n}\cdot\mathbf{e})^2\mathbf{a} + \mathbf{Q}(\mathbf{n})\mathbf{a} = \rho U^2\mathbf{a}, \tag{3.12}$$

where $\mathbf{Q}(\mathbf{n})$ is the acoustic tensor (2.9) corresponding to the deformation gradient $\mathbf{F} = \mathbf{1}$.

By (3.10), we must consider two cases: $\mathbf{n}\cdot\mathbf{e} = 0$ and $\mathbf{a}\cdot\mathbf{e} = 0$.

Case 1

$$\mathbf{n}\cdot\mathbf{e} = 0. \tag{3.13}$$

Here trivially,

$$\mathbf{Q}(\mathbf{n})\mathbf{a} = \rho U^2\mathbf{a}. \tag{3.14}$$

Case 2

$$\mathbf{a}\cdot\mathbf{e} = 0. \tag{3.15}$$

If we take the inner product of (3.12) with **e**, we find that

$$\frac{iU(\mathbf{n}\cdot\mathbf{e})\alpha}{\omega} = \mathbf{e}\cdot\mathbf{Q}(\mathbf{n})\mathbf{a}, \tag{3.16}$$

and hence (3.12) implies that

$$\mathbf{P}\mathbf{Q}(\mathbf{n})\mathbf{a} = [\rho U^2 - \sigma_0(\mathbf{n}\cdot\mathbf{e})^2]\mathbf{a}, \tag{3.17}$$

where now

$$\mathbf{P} = \mathbf{1} - \mathbf{e}\otimes\mathbf{e} \tag{3.18}$$

is the projection onto the plane perpendicular to **e**.

When comparing these results with those of the previous section one should bear in mind that in section 2 the material ahead of the wave was allowed to have an arbitrary deformation gradient **F**, while the progressive waves studied here are propagating through material at rest in the reference configuration. If in our results on acceleration waves we assume that

$\mathbf{F} = \mathbf{1}$ at the wave, then the results of the two sections are comparable, and, in fact, (2.4), (2.11), (2.15), and (2.16) are identical to (3.10), (3.14), (3.17), and (3.18), respectively.

As in the case of an acceleration wave, when the underlying reaction stress σ_0 is zero, (3.17) reduces to

$$\mathbf{PQ}(\mathbf{n})\mathbf{a}_0 = \rho U_0{}^2 \mathbf{a}_0, \tag{3.19}$$

and, as before, if $(\mathbf{a}_0, U_0)$ is a solution of (3.19), then

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_0 \\ \rho U^2 &= \rho U_0{}^2 + \sigma_0(\mathbf{n} \cdot \mathbf{e})^2 \end{aligned} \tag{3.20}$$

is a solution of (3.17) for arbitary σ_0. Thus, when $\mathbf{n} \cdot \mathbf{e} \neq 0$, U will be imaginary when

$$\sigma_0 < 0, \qquad |\sigma_0| > \frac{\rho U_0{}^2}{(\mathbf{n} \cdot \mathbf{e})^2}. \tag{3.21}$$

Therefore, in view of (3.7), when (3.21) is satisfied both $\mathbf{u}$ and σ tend to infinity exponentially as $t \to \infty$. Further, for

$$\sigma_0 = -\frac{\rho U_0{}^2}{(\mathbf{n} \cdot \mathbf{e})^2}, \tag{3.22}$$

we have

$$U = 0. \tag{3.23}$$

For $U = 0$ the equations (3.7) make no sense. If, however, we replace the term ω/U in (3.7) by $1/\lambda$ and repeat the analysis we find that

$$\begin{aligned} \mathbf{u} &= \mathbf{a} \exp\left\{\frac{i\mathbf{p} \cdot \mathbf{n}}{\lambda}\right\}, \\ \sigma - \sigma_0 &= \alpha \exp\left\{\frac{i\mathbf{p} \cdot \mathbf{n}}{\lambda}\right\} \end{aligned} \tag{3.24}$$

when (3.22) holds. These represent time-independent "buckied mode shapes" for the material. In (3.24) the wave-length λ is arbitrary; thus there exists a buckled mode shape with every possible wave-length.

4. CONCLUDING REMARKS

We have also derived, using the standard method of analysis, an explicit expression for the amplitude of a plane acceleration wave propagating through an inextensible elastic body which is at rest in a homogeneous configuration. This expression is independent of the underlying reaction stress σ and leads to results which are completely analogous to those for an unconstrained body.*

The analysis given in this paper could also be applied to elastic bodies subject to the constraint†

$$\mathbf{Fe}_1 \cdot \mathbf{Fe}_2 = 0, \tag{4.1}$$

*Cf. Chen [6], §§6, 7.

†Cf. e.g. Gurtin and Podio Guidugli [4].

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors. For this type of body the reaction stress is a pure shear of the form $\tau/2(\mathbf{Fe}_1 \otimes \mathbf{e}_2 + \mathbf{Fe}_2 \otimes \mathbf{e}_1)$, and the results obtained are completely analogous to those of sections 2 and 3 for inextensible bodies, except that the instability is now one of shear rather than compression. Indeed, as before, there are two cases of interest: case 1, in which $\mathbf{n}$ is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, and case 2, in which

$$\mathbf{g} = (\mathbf{e}_1 \cdot \mathbf{n})\mathbf{Fe}_2 + (\mathbf{e}_2 \cdot \mathbf{n})\mathbf{Fe}_1 \tag{4.2}$$

is perpendicular to $\mathbf{a}$. In case 1 the Fresnel–Hadamard condition (2.11) holds, while in case 2 the propagation condition for both acceleration waves and infinitesimal progressive waves is

$$\mathbf{PQ}(\mathbf{n})\mathbf{a} = [\rho U^2 - \tau(\mathbf{n} \cdot \mathbf{e}_1)(\mathbf{n} \cdot \mathbf{e}_2)]\mathbf{a}, \tag{4.3}$$

with

$$\mathbf{P} = \mathbf{1} - \frac{\mathbf{g} \otimes \mathbf{g}}{|\mathbf{g}|^2}, \tag{4.4}$$

provided, of course, we set $\mathbf{F} = \mathbf{1}$ for progressive waves.

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Резюме — В этой работе рассматривается распространение волн ускорения в нерастяжимых эластичных телах. Хотя, расчеты эти являются только упражнением, результаты интересны и совершенно не похожи на соответствующие результаты по не стесненным связями телам. В самом деле, волна двигающаяся в не растяжимом направлении по необходимости должна быть поперечной, и, если напряжение реакции сжимающее и достаточно сильное, соответствующая скорость распространения становится нереальной, так что даже поперечные волны больше не существуют.

Также рассматривались (бесконечно малые) прогрессивные волны, и нашли, что условия соответствуют условиям волн ускорения. В этом случае, однако, нереальные скорости распространения имеют определенное физическое значение: они означают исключительный рост волн. Особенно, когда напряжение реакции сжимающее и сильное, тогда поперечная прогрессивная волна передвигающаяся по нерастяжимому направлению неограниченно растет. Предполагается, что присутствует местный продольный изгиб.